Aicha Benkartab; Ahmed Mohammed Cherif
Deformations of Metrics and Biharmonic Maps

Communications in Mathematics, Vol. 28 (2020), No. 3, 263–275

Persistent URL: `http://dml.cz/dmlcz/148707`

Communications in Mathematics 28 (2020) 263–275
DOI: 10.2478/cm-2020-0022

Deformations of Metrics and Biharmonic Maps

Aicha Benkartab, Ahmed Mohammed Cherif

Abstract. We construct biharmonic non-harmonic maps between Riemannian manifolds (M, g) and (N, h) by first making the ansatz that $\varphi\colon (M, g) \to (N, h)$ be a harmonic map and then deforming the metric on N by

$$\tilde{h}_\alpha = \alpha h + (1 - \alpha)\mathrm{d}f \otimes \mathrm{d}f$$

to render φ biharmonic, where f is a smooth function with gradient of constant norm on (N, h) and $\alpha \in (0, 1)$. We construct new examples of biharmonic non-harmonic maps, and we characterize the biharmonicity of some curves on Riemannian manifolds.

1 Introduction

Let (M, g) and (N, h) be two Riemannian manifolds. The energy functional of a map $\varphi \in C^\infty(M, N)$ is defined by

$$E(\varphi) = \frac{1}{2}\int_M |\mathrm{d}\varphi|^2 v^g\,, \tag{1}$$

where $|\mathrm{d}\varphi|$ is the Hilbert-Schmidt norm of the differential $\mathrm{d}\varphi$ and v^g is the volume element on (M, g). A map $\varphi \in C^\infty(M, N)$ is called harmonic if it is a critical point of the energy functional, that is, if it is a solution of the Euler Lagrange equation associated to (1)

$$\tau(\varphi) = \operatorname{trace} \nabla \mathrm{d}\varphi = \nabla^\varphi_{e_i} \mathrm{d}\varphi(e_i) - \mathrm{d}\varphi(\nabla^M_{e_i} e_i) = 0\,, \tag{2}$$

where $\{e_i\}_{i=1}^m$ is an orthonormal frame on (M, g), $m = \dim M$, ∇^M is the Levi--Civita connection of (M, g), and ∇^φ denote the pull-back connection on $\varphi^{-1}TN$.

2020 MSC: 53C20, 58E20, 53C22
Key words: Riemannian geometry, Harmonic maps, Biharmonic maps
Affiliation:
Aicha Benkartab – Mascara University, Faculty of Science and technology, Mascara 29000, Algeria
E-mail: benkartab.aicha@univ-mascara.dz
Ahmed Mohammed Cherif – Mascara University, Faculty of Exact Sciences, Mascara 29000, Algeria
E-mail: a.mohammedcherif@univ-mascara.dz

Harmonic maps are solutions of a second order nonlinear elliptic system and they play a very important role in many branches of mathematics and physics where they may serve as a model for liquid crystal (see [9]). One can refer to [6], [7], [8] for background on harmonic maps. A natural generalization of harmonic maps is given by integrating the square of the norm of the tension field. More precisely, the bi-energy functional of a map $\varphi \in C^\infty(M,N)$ is defined by

$$E_2(\varphi) = \frac{1}{2}\int_M |\tau(\varphi)|^2 v^g \,. \tag{3}$$

A map $\varphi \in C^\infty(M,N)$ is called biharmonic if it is a critical point of the bi-energy functional, that is, if it is a solution of the Euler Lagrange equation associated to (3)

$$\begin{aligned}\tau_2(\varphi) &= -\operatorname{trace} R^N(\tau(\varphi), \mathrm{d}\varphi)\mathrm{d}\varphi - \operatorname{trace}(\nabla^\varphi)^2\tau(\varphi)\\ &= -R^N(\tau(\varphi), \mathrm{d}\varphi(e_i))\mathrm{d}\varphi(e_i) - \nabla^\varphi_{e_i}\nabla^\varphi_{e_i}\tau(\varphi) + \nabla^\varphi_{\nabla^M_{e_i}e_i}\tau(\varphi) = 0\,,\end{aligned} \tag{4}$$

where R^N is the curvature tensor of (N,h) (see [5], [12]). Clearly, harmonic maps are biharmonic. G.Y. Jiang [12] proved that if M is compact without boundary and the sectional curvature of (N,h) is negative, then any biharmonic map $\varphi \in C^\infty(M,N)$ is harmonic. So it is interesting to construct biharmonic non-harmonic maps. We refer the reader to [2], [5], [10], [11] for other examples and different approaches to their construction.

In this paper, we deform the codomain metric by $\tilde{h}_\alpha = \alpha h + (1-\alpha)\mathrm{d}f \otimes \mathrm{d}f$, where $\alpha \in (0,1)$ and $f \in C^\infty(N)$, in order to render a map biharmonic non-harmonic with respect to the new metric, we give a necessary and sufficient condition on f and α such that $\varphi\colon (M,g) \to (N,\tilde{h}_\alpha)$ is biharmonic non-harmonic. So by suitable choices of f, we are able to give new examples of biharmonic non-harmonic maps.

2 Deformations of Metrics

Let M be a Riemannian manifold equipped with Riemannian metric g, and f a smooth function on M. We define on M a Riemannian metric, denoted $\tilde{g}_\alpha$, by

$$\tilde{g}_\alpha = \alpha g + (1-\alpha)\mathrm{d}f \otimes \mathrm{d}f\,,$$

for some constant $\alpha \in (0,1)$. In the seminal work [4], we obtain the following results.

Theorem 1. *Let (M,g) be a Riemannian manifold and $\widetilde{\nabla}$ denote the Levi-Civita connection of $(M,\tilde{g}_\alpha)$. Then*

$$\widetilde{\nabla}_X Y = \nabla_X Y + \frac{(1-\alpha)\operatorname{Hess}_f(X,Y)}{\alpha + (1-\alpha)\|\operatorname{grad} f\|^2}\operatorname{grad} f\,,$$

where ∇ is the Levi-Civita connection of (M,g), Hess_f (resp. $\operatorname{grad} f$) is the Hessian (resp. the gradient vector) of f with respect to g, and

$$\|\operatorname{grad} f\|^2 = g(\operatorname{grad} f, \operatorname{grad} f)\,.$$

Proof. Let $X, Y, Z \in \Gamma(TM)$. From the Koszul formula (see [13]), we have

$$\begin{aligned} 2\tilde{g}_\alpha(\widetilde{\nabla}_X Y, Z) = 2\alpha g(\nabla_X Y, Z) + (1-\alpha)\Big\{ & X(Y(f)Z(f)) + Y(Z(f)X(f)) \\ & - Z(X(f)Y(f)) + Z(f)[X,Y](f) + Y(f)[Z,X](f) \\ & - X(f)[Y,Z](f)\Big\} . \end{aligned} \tag{5}$$

Let $\{e_i\}_{i=1}^m$ be a geodesic frame on (M, g) at $x \in M$ (see [3]), where $m = \dim M$. By (5) we obtain

$$\begin{aligned} 2\tilde{g}_\alpha(\widetilde{\nabla}_X Y, e_i) = 2\alpha g(\nabla_X Y, e_i) + (1-\alpha)\Big\{ & X(Y(f)g(e_i, \operatorname{grad} f)) \\ & + Y(X(f)g(e_i, \operatorname{grad} f)) - e_i(g(X, \operatorname{grad} f)g(Y, \operatorname{grad} f)) \\ & + e_i(f)[X,Y](f) + Y(f)(\nabla_{e_i} X)(f) + X(f)(\nabla_{e_i} Y)(f)\Big\} , \end{aligned} \tag{6}$$

from equation (6), and the definition of Hessian (see [13]), we get

$$\begin{aligned} \tilde{g}_\alpha(\widetilde{\nabla}_X Y, e_i) = {} & \alpha g(\nabla_X Y, e_i) + (1-\alpha) g(\nabla_X Y, \operatorname{grad} f) g(e_i, \operatorname{grad} f) \\ & + (1-\alpha) \operatorname{Hess}_f(X, Y) g(e_i, \operatorname{grad} f) , \end{aligned} \tag{7}$$

from equation (7), we obtain

$$\begin{aligned} \tilde{g}_\alpha(\widetilde{\nabla}_X Y, Z) = {} & \alpha g(\nabla_X Y, Z) + (1-\alpha) g(\nabla_X Y, \operatorname{grad} f) g(Z, \operatorname{grad} f) \\ & + (1-\alpha) \operatorname{Hess}_f(X, Y) g(Z, \operatorname{grad} f) , \end{aligned} \tag{8}$$

by the definition of the Riemannian metric $\tilde{g}_\alpha$, and (8) we find that

$$\tilde{g}_\alpha(\widetilde{\nabla}_X Y, Z) = \tilde{g}_\alpha(\nabla_X Y, Z) + (1-\alpha) \operatorname{Hess}_f(X, Y) Z(f) . \tag{9}$$

Hence Theorem 1 follows from (9), with the following

$$Z(f) = \frac{1}{\alpha + (1-\alpha)\|\operatorname{grad} f\|^2} \tilde{g}_\alpha(Z, \operatorname{grad} f) . \qquad \square$$

Now consider the curvature tensor $\widetilde{R}$ of $(M, \tilde{g}_\alpha)$, writing R for the curvature tensor of (M, g). We have the following result:

Theorem 2. *For all $X, Y, Z \in \Gamma(TM)$, we have*

$$\begin{aligned} \widetilde{R}(X,Y)Z = R(X,Y)Z & + \frac{(1-\alpha) g(R(X,Y) \operatorname{grad} f, Z)}{\alpha + (1-\alpha)\|\operatorname{grad} f\|^2} \operatorname{grad} f \\ & - \frac{(1-\alpha)^2 \operatorname{Hess}_f(X, \operatorname{grad} f) \operatorname{Hess}_f(Y, Z)}{(\alpha + (1-\alpha)\|\operatorname{grad} f\|^2)^2} \operatorname{grad} f \\ & + \frac{(1-\alpha)^2 \operatorname{Hess}_f(Y, \operatorname{grad} f) \operatorname{Hess}_f(X, Z)}{(\alpha + (1-\alpha)\|\operatorname{grad} f\|^2)^2} \operatorname{grad} f \\ & + \frac{(1-\alpha) \operatorname{Hess}_f(Y, Z)}{\alpha + (1-\alpha)\|\operatorname{grad} f\|^2} \nabla_X \operatorname{grad} f \\ & - \frac{(1-\alpha) \operatorname{Hess}_f(X, Z)}{\alpha + (1-\alpha)\|\operatorname{grad} f\|^2} \nabla_Y \operatorname{grad} f . \end{aligned}$$

Proof. By the definition of the curvature tensor $\widetilde{R}$,

$$\widetilde{R}(X,Y)Z = \widetilde{\nabla}_X \widetilde{\nabla}_Y Z - \widetilde{\nabla}_Y \widetilde{\nabla}_X Z - \widetilde{\nabla}_{[X,Y]} Z\,,$$

and Theorem 1 we obtain

$$\begin{aligned}
\widetilde{R}(X,Y)Z = {} & \widetilde{\nabla}_X \left(\nabla_Y Z + \frac{(1-\alpha)\operatorname{Hess}_f(Y,Z)}{\alpha + (1-\alpha)\|\operatorname{grad} f\|^2} \operatorname{grad} f \right) \\
& - \widetilde{\nabla}_Y \left(\nabla_X Z + \frac{(1-\alpha)\operatorname{Hess}_f(X,Z)}{\alpha + (1-\alpha)\|\operatorname{grad} f\|^2} \operatorname{grad} f \right) \\
& - \left(\nabla_{[X,Y]} Z + \frac{(1-\alpha)\operatorname{Hess}_f([X,Y],Z)}{\alpha + (1-\alpha)\|\operatorname{grad} f\|^2} \operatorname{grad} f \right),
\end{aligned} \tag{10}$$

the first term of (10) is given by

$$\begin{aligned}
& \widetilde{\nabla}_X \left(\nabla_Y Z + \frac{(1-\alpha)\operatorname{Hess}_f(Y,Z)}{\alpha + (1-\alpha)\|\operatorname{grad} f\|^2} \operatorname{grad} f \right) \\
& \quad = \nabla_X \left(\nabla_Y Z + \frac{(1-\alpha)\operatorname{Hess}_f(Y,Z)}{\alpha + (1-\alpha)\|\operatorname{grad} f\|^2} \operatorname{grad} f \right) \\
& \qquad + \frac{(1-\alpha)\operatorname{Hess}_f \left(X, \nabla_Y Z + \frac{(1-\alpha)\operatorname{Hess}_f(Y,Z)}{\alpha+(1-\alpha)\|\operatorname{grad} f\|^2} \operatorname{grad} f \right)}{\alpha + (1-\alpha)\|\operatorname{grad} f\|^2} \operatorname{grad} f\,,
\end{aligned} \tag{11}$$

by equation (11), and the definition of Hessian, we obtain

$$\begin{aligned}
& \widetilde{\nabla}_X \left(\nabla_Y Z + \frac{(1-\alpha)\operatorname{Hess}_f(Y,Z)}{\alpha + (1-\alpha)\|\operatorname{grad} f\|^2} \operatorname{grad} f \right) \\
& \quad = \nabla_X \nabla_Y Z + \frac{(1-\alpha) g(\nabla_X \nabla_Y \operatorname{grad} f, Z)}{\alpha + (1-\alpha)\|\operatorname{grad} f\|^2} \operatorname{grad} f \\
& \qquad + \frac{(1-\alpha)\operatorname{Hess}_f(Y, \nabla_X Z)}{\alpha + (1-\alpha)\|\operatorname{grad} f\|^2} \operatorname{grad} f \\
& \qquad - \frac{(1-\alpha)^2 \operatorname{Hess}_f(X, \operatorname{grad} f)\operatorname{Hess}_f(Y,Z)}{(\alpha + (1-\alpha)\|\operatorname{grad} f\|^2)^2} \operatorname{grad} f \\
& \qquad + \frac{(1-\alpha)\operatorname{Hess}_f(Y,Z)}{\alpha + (1-\alpha)\|\operatorname{grad} f\|^2} \nabla_X \operatorname{grad} f \\
& \qquad + \frac{(1-\alpha)\operatorname{Hess}_f(X, \nabla_Y Z)}{\alpha + (1-\alpha)\|\operatorname{grad} f\|^2} \operatorname{grad} f\,.
\end{aligned} \tag{12}$$

Using the similar method, the second term of (10) is given by

$$\begin{aligned}
-\widetilde{\nabla}_Y\Big(\nabla_X Z + \frac{(1-\alpha)\operatorname{Hess}_f(X,Z)}{\alpha+(1-\alpha)\|\operatorname{grad} f\|^2}\operatorname{grad} f\Big) \\
&= -\nabla_Y\nabla_X Z - \frac{(1-\alpha)g(\nabla_Y\nabla_X \operatorname{grad} f, Z)}{\alpha+(1-\alpha)\|\operatorname{grad} f\|^2}\operatorname{grad} f \\
&\quad - \frac{(1-\alpha)\operatorname{Hess}_f(X,\nabla_Y Z)}{\alpha+(1-\alpha)\|\operatorname{grad} f\|^2}\operatorname{grad} f \\
&\quad + \frac{(1-\alpha)^2\operatorname{Hess}_f(Y,\operatorname{grad} f)\operatorname{Hess}_f(X,Z)}{(\alpha+(1-\alpha)\|\operatorname{grad} f\|^2)^2}\operatorname{grad} f \\
&\quad - \frac{(1-\alpha)\operatorname{Hess}_f(X,Z)}{\alpha+(1-\alpha)\|\operatorname{grad} f\|^2}\nabla_Y \operatorname{grad} f \\
&\quad - \frac{(1-\alpha)\operatorname{Hess}_f(Y,\nabla_X Z)}{\alpha+(1-\alpha)\|\operatorname{grad} f\|^2}\operatorname{grad} f. \qquad (13)
\end{aligned}$$

Theorem 2 follows from equations (10), (12) and (13). □

3 The biharmonicity of $\varphi\colon (M,g)\to(N,\tilde{h}_\alpha)$

We now consider the effects of a deformation of the codomain metric, as regards harmonic and biharmonic mappings.

Theorem 3. *Let $\varphi\colon (M,g)\to(N,h)$ be a harmonic map between two Riemannian manifolds and let the Riemannian metric $\tilde{h}_\alpha=\alpha h+(1-\alpha)\mathrm{d}f\otimes\mathrm{d}f$, where $\alpha\in(0,1)$ and $f\in C^\infty(N)$. We suppose that $\|\operatorname{grad} f\|=1$. If the function $\Delta(f\circ\varphi)$ is a non--null constant on M, then the map $\varphi\colon (M,g)\to(N,\tilde{h}_\alpha)$ is proper biharmonic if and only if the gradient vector of f is Jacobi field along φ, i.e. $(\operatorname{grad} f)\circ\varphi\in\ker J_\varphi$ where J_φ is a Jacobi operator corresponding to φ.*

Proof. Let $\{e_i\}_{i=1}^m$ be a normal orthonormal frame on (M,g) at x, where $m=\dim M$. Then the map $\varphi\colon (M,g)\to(N,\tilde{h}_\alpha)$ is biharmonic if and only if

$$\tilde{\tau}_2(\varphi) = -\tilde{R}^N(\tilde{\tau}(\varphi),\mathrm{d}\varphi(e_i))\mathrm{d}\varphi(e_i) - \tilde{\nabla}^\varphi_{e_i}\tilde{\nabla}^\varphi_{e_i}\tilde{\tau}(\varphi) = 0\,, \qquad (14)$$

where $\tilde{R}^N$ is the Riemannian curvature with respect to $\tilde{h}_\alpha$, $\tilde{\tau}(\varphi)$ denotes the tension field of the map φ with respect to $\tilde{h}_\alpha$, and $\tilde{\nabla}^\varphi$ is the pull-back connection with respect to the metric $\tilde{h}_\alpha$. First, we compute the tension field $\tilde{\tau}(\varphi)$,

$$\begin{aligned}
\tilde{\tau}(\varphi) &= \tilde{\nabla}^\varphi_{e_i}\mathrm{d}\varphi(e_i) = \widetilde{\nabla}^N_{\mathrm{d}\varphi(e_i)}\mathrm{d}\varphi(e_i) \\
&= \tau(\varphi) + \frac{(1-\alpha)\operatorname{Hess}_f(\mathrm{d}\varphi(e_i),\mathrm{d}\varphi(e_i))}{\alpha+(1-\alpha)\|\operatorname{grad} f\|^2\circ\varphi}(\operatorname{grad} f)\circ\varphi \\
&= (1-\alpha)\operatorname{Hess}_f(\mathrm{d}\varphi(e_i),\mathrm{d}\varphi(e_i))(\operatorname{grad} f)\circ\varphi,
\end{aligned}$$

since $\Delta(f\circ\varphi)=\mathrm{d}f(\tau(\varphi))+\operatorname{trace}\operatorname{Hess}_f(\mathrm{d}\varphi,\mathrm{d}\varphi)$ (see [3]), and $\tau(\varphi)=0$, we have $\tilde{\tau}(\varphi)=\lambda(\operatorname{grad} f)\circ\varphi$, with $\lambda=(1-\alpha)\Delta(f\circ\varphi)$ is a non-null constant. Now, we compute the first term of (14), from Theorem 2, we have

$$\begin{aligned}
&\widetilde{R}^N(\tilde{\tau}(\varphi), \mathrm{d}\varphi(e_i))\mathrm{d}\varphi(e_i) \\
&\qquad = \lambda\Big\{R^N(\operatorname{grad} f, \mathrm{d}\varphi(e_i))\mathrm{d}\varphi(e_i) \\
&\qquad\quad + \frac{(1-\alpha)h(R^N(\operatorname{grad} f, \mathrm{d}\varphi(e_i))\operatorname{grad} f, \mathrm{d}\varphi(e_i))}{\alpha+(1-\alpha)\|\operatorname{grad} f\|^2}\operatorname{grad} f \\
&\qquad\quad - \frac{(1-\alpha)^2 \operatorname{Hess}_f(\operatorname{grad} f, \operatorname{grad} f)\operatorname{Hess}_f(\mathrm{d}\varphi(e_i), \mathrm{d}\varphi(e_i))}{(\alpha+(1-\alpha)\|\operatorname{grad} f\|^2)^2}\operatorname{grad} f \\
&\qquad\quad + \frac{(1-\alpha)^2 \operatorname{Hess}_f(\mathrm{d}\varphi(e_i), \operatorname{grad} f)\operatorname{Hess}_f(\operatorname{grad} f, \mathrm{d}\varphi(e_i))}{(\alpha+(1-\alpha)\|\operatorname{grad} f\|^2)^2}\operatorname{grad} f \\
&\qquad\quad + \frac{(1-\alpha)\operatorname{Hess}_f(\mathrm{d}\varphi(e_i), \mathrm{d}\varphi(e_i))}{\alpha+(1-\alpha)\|\operatorname{grad} f\|^2}\nabla^N_{\operatorname{grad} f}\operatorname{grad} f \\
&\qquad\quad - \frac{(1-\alpha)\operatorname{Hess}_f(\operatorname{grad} f, \mathrm{d}\varphi(e_i))}{\alpha+(1-\alpha)\|\operatorname{grad} f\|^2}\nabla^N_{\mathrm{d}\varphi(e_i)}\operatorname{grad} f\Big\}\circ\varphi\,, \qquad (15)
\end{aligned}$$

since $\|\operatorname{grad} f\| = 1$, is constant on N, we obtain

$$\operatorname{Hess}_f(\operatorname{grad} f, X) = 0, \quad \nabla^N_{\operatorname{grad} f}\operatorname{grad} f = \frac{1}{2}\operatorname{grad}\|\operatorname{grad} f\|^2 = 0, \qquad (16)$$

for all $X \in \Gamma(TN)$, the equation (15) becomes

$$\begin{aligned}
&\widetilde{R}^N(\tilde{\tau}(\varphi), \mathrm{d}\varphi(e_i))\mathrm{d}\varphi(e_i) \\
&\qquad = \lambda\Big\{R^N(\operatorname{grad} f, \mathrm{d}\varphi(e_i))\mathrm{d}\varphi(e_i) \\
&\qquad\quad + (1-\alpha)h(R^N(\operatorname{grad} f, \mathrm{d}\varphi(e_i))\operatorname{grad} f, \mathrm{d}\varphi(e_i))\operatorname{grad} f\Big\}\circ\varphi\,. \qquad (17)
\end{aligned}$$

The second term of (14) is given by

$$\begin{aligned}
\tilde{\nabla}^\varphi_{e_i}\tilde{\nabla}^\varphi_{e_i}\tilde{\tau}(\varphi) &= \lambda\tilde{\nabla}^\varphi_{e_i}\tilde{\nabla}^\varphi_{e_i}(\operatorname{grad} f)\circ\varphi \\
&= \lambda\tilde{\nabla}^\varphi_{e_i}(\tilde{\nabla}^N_{\mathrm{d}\varphi(e_i)}\operatorname{grad} f)\circ\varphi \\
&= \lambda\tilde{\nabla}^\varphi_{e_i}\Big\{(\nabla^N_{\mathrm{d}\varphi(e_i)}\operatorname{grad} f)\circ\varphi \\
&\quad + \frac{(1-\alpha)\operatorname{Hess}_f(\mathrm{d}\varphi(e_i), (\operatorname{grad} f)\circ\varphi)}{\alpha+(1-\alpha)\|\operatorname{grad} f\|^2\circ\varphi}(\operatorname{grad} f)\circ\varphi\Big\}, \qquad (18)
\end{aligned}$$

from equations (16) and (18), we find that

$$\begin{aligned}
\tilde{\nabla}^\varphi_{e_i}\tilde{\nabla}^\varphi_{e_i}\tilde{\tau}(\varphi) &= \lambda\nabla^\varphi_{e_i}\nabla^\varphi_{e_i}(\operatorname{grad} f)\circ\varphi \\
&\quad + (1-\alpha)\lambda\operatorname{Hess}_f(\mathrm{d}\varphi(e_i), \nabla^\varphi_{e_i}(\operatorname{grad} f)\circ\varphi)(\operatorname{grad} f)\circ\varphi\,, \qquad (19)
\end{aligned}$$

and note that

$$\operatorname{Hess}_f(\mathrm{d}\varphi(e_i), \nabla^\varphi_{e_i}(\operatorname{grad} f)\circ\varphi) = -h((\operatorname{grad} f)\circ\varphi, \nabla^\varphi_{e_i}\nabla^\varphi_{e_i}(\operatorname{grad} f)\circ\varphi)\,.$$

So, the map $\varphi\colon (M,g) \to (N,\tilde{h}_\alpha)$ is biharmonic if and only if

$$J_\varphi\big((\operatorname{grad} f)\circ\varphi\big) - (1-\alpha)h\big(J_\varphi\big((\operatorname{grad} f)\circ\varphi\big),(\operatorname{grad} f)\circ\varphi\big)(\operatorname{grad} f)\circ\varphi = 0\,. \quad (20)$$

Note that, the equation (20) is equivalent to $J_\varphi\big((\operatorname{grad} f)\circ\varphi\big) = 0$. □

Example 1. Let $M = \mathbb{R}^2$ and $N = \mathbb{H}^2 = \{(y_1,y_2) \in \mathbb{R}^2 | y_2 > 0\}$. We consider the harmonic map $\varphi\colon (M, \mathrm{d}x_1^2 + \mathrm{d}x_2^2) \to (N, y_2^2(\mathrm{d}y_1^2 + \mathrm{d}y_2^2))$, $(x_1,x_2) \mapsto (x_1, \sqrt{x_2^2+1})$, and let the function $f(y_1,y_2) = \frac{1}{2}y_2^2$. A straightforward calculation shows that $\|\operatorname{grad} f\| = 1$, $\Delta(f\circ\varphi) = 1$, $(\operatorname{grad} f)\circ\varphi = \Big(0, \frac{1}{\sqrt{x_2^2+1}}\Big)$ and $J_\varphi((\operatorname{grad} f)\circ\varphi) = 0$. Thus, with respect to metric $\tilde{h}_\alpha = y_2^2(\alpha\mathrm{d}y_1^2 + \mathrm{d}y_2^2)$, the map φ is biharmonic non-harmonic, with $\tilde{\tau}(\varphi) = \Big(0, \frac{1-\alpha}{\sqrt{x_2^2+1}}\Big)$.

Remark 1.
- Let $\varphi\colon (M,g) \to (N,h)$ be a harmonic map between two Riemannian manifolds and $\tilde{h}_\alpha = \alpha h + (1-\alpha)\mathrm{d}f \otimes \mathrm{d}f$, where $\alpha \in (0,1)$ and $f \in C^\infty(N)$ such that $\|\operatorname{grad} f\| = 1$. Then the map $\varphi\colon (M,g) \to (N,\tilde{h}_\alpha)$ is harmonic if and only if $f\circ\varphi$ is harmonic on (M,g).
- Let (M,g) be a Riemannian manifold, and let f be a smooth function on M such that $\|\operatorname{grad} f\| = 1$ and $\Delta f = k$, where $k \in \mathbb{R}$. Then, the identity map from (M,g) to $(M,\tilde{g}_\alpha)$ is biharmonic if and only if it is harmonic. Indeed; from Theorem 3 the identity map from (M,g) to $(M,\tilde{g}_\alpha)$ is a biharmonic map if and only if $\operatorname{Ricci}(\operatorname{grad} f) = 0$, and by Bochner-Weitzenböck formula for smooth functions (see [14])
$$\frac{1}{2}\Delta(\|\operatorname{grad} f\|^2) = \|\operatorname{Hess}_f\|^2 + g(\operatorname{grad} f, \operatorname{grad}(\Delta f)) + \operatorname{Ric}(\operatorname{grad} f, \operatorname{grad} f)\,,$$
we obtain $\|\operatorname{Hess}_f\| = 0$, so that $\Delta f = 0$, that is the identity map from (M,g) to $(M,\tilde{g}_\alpha)$ is harmonic map.

4 The biharmonicity of the identity map $(M,\tilde{g}_\alpha) \to (M,\tilde{g}_\beta)$

Let (M,g) be a Riemannian manifold, $f \in C^\infty(M)$, $\alpha,\beta \in (0,1)$, and denote by

$$\begin{aligned}\widetilde{I}_{\alpha,\beta}\colon (M,\tilde{g}_\alpha) &\to (M,\tilde{g}_\beta)\,,\\ x &\mapsto x\end{aligned}$$

the identity map, where $\tilde{g}_\alpha = \alpha g + (1-\alpha)\mathrm{d}f\otimes\mathrm{d}f$ and $\tilde{g}_\beta = \beta g + (1-\beta)\mathrm{d}f\otimes\mathrm{d}f$.

Theorem 4. *If $\alpha \neq \beta$, and $\|\operatorname{grad} f\| = 1$. Then the identity map $\widetilde{I}_{\alpha,\beta}$ is a proper biharmonic if and only if the function f is non-harmonic on M, and satisfying the following*

$$\begin{aligned}2\Delta f\operatorname{Ricci}(\operatorname{grad} f) = &-\frac{1}{\beta}\Delta^2 f\operatorname{grad} f - 2\nabla_{\operatorname{grad}\Delta f}\operatorname{grad} f - \Delta f\operatorname{grad}\Delta f\\ &+\frac{1-\alpha}{\beta}\Delta f g(\operatorname{grad} f,\operatorname{grad}\Delta f)\operatorname{grad} f\\ &+\frac{1-\alpha}{\beta}\operatorname{Hess}_{\Delta f}(\operatorname{grad} f,\operatorname{grad} f)\operatorname{grad} f\,,\end{aligned}$$

where Δf is the Laplacian of f with respect to g, and $\Delta^2 f = \Delta(\Delta f)$.

Proof. Let $\{e_i\}_{i=1}^m$ be an orthonormal frame on M with respect to the metric g, such that $e_1 = \operatorname{grad} f$, it is easy to prove that $\left\{e_1, \frac{1}{\sqrt{\alpha}}e_i\right\}_{i=2}^m$ is a orthonormal frame on M with respect to the metric $\tilde{g}_\alpha$, where $m = \dim M$. Let $\widetilde{\nabla}^\alpha$ (resp. $\widetilde{\nabla}^\beta$) the Levi-Civita connection of $(M, \tilde{g}_\alpha)$ (resp. of $(M, \tilde{g}_\beta)$), then the tension field of $\widetilde{I}_{\alpha,\beta}$ is given by

$$\begin{aligned}\tau(\widetilde{I}_{\alpha,\beta}) &= \nabla^{\widetilde{I}_{\alpha,\beta}}_{e_1} \mathrm{d}\widetilde{I}_{\alpha,\beta}(e_1) - \mathrm{d}\widetilde{I}_{\alpha,\beta}(\widetilde{\nabla}^\alpha_{e_1} e_1) + \frac{1}{\alpha}\sum_{i=2}^m \left\{\nabla^{\widetilde{I}_{\alpha,\beta}}_{e_i} \mathrm{d}\widetilde{I}_{\alpha,\beta}(e_i) - \mathrm{d}\widetilde{I}_{\alpha,\beta}(\widetilde{\nabla}^\alpha_{e_i} e_i)\right\} \\ &= \widetilde{\nabla}^\beta_{e_1} e_1 - \widetilde{\nabla}^\alpha_{e_1} e_1 + \frac{1}{\alpha}\sum_{i=2}^m \left\{\widetilde{\nabla}^\beta_{e_i} e_i - \widetilde{\nabla}^\alpha_{e_i} e_i\right\},\end{aligned}$$

using Theorem 1, with $\|\operatorname{grad} f\| = 1$, we have

$$\tau(\widetilde{I}_{\alpha,\beta}) = \frac{\alpha-\beta}{\alpha}\sum_{i=2}^m \operatorname{Hess}_f(e_i, e_i)\operatorname{grad} f\,, \tag{21}$$

since $\operatorname{Hess}_f(e_1, e_1) = 0$, the equation (21) becomes

$$\tau(\widetilde{I}_{\alpha,\beta}) = \frac{\alpha-\beta}{\alpha}\Delta f \operatorname{grad} f\,.$$

Note that $\widetilde{I}_{\alpha,\beta}$ is harmonic if and only if $\Delta f = 0$, i.e. the function f is harmonic on (M, g). We compute the bitension field of the identity $\widetilde{I}_{\alpha,\beta}$, for all $i = 1, \dots, m$ we have

$$\widetilde{R}_\beta(\tau(\widetilde{I}_{\alpha,\beta}), \mathrm{d}\widetilde{I}_{\alpha,\beta}(e_i))\mathrm{d}\widetilde{I}_{\alpha,\beta}(e_i) = \frac{\alpha-\beta}{\alpha}\Delta f \widetilde{R}_\beta(\operatorname{grad} f, e_i)e_i\,, \tag{22}$$

where $\widetilde{R}_\beta$ is the curvature tensor of $\widetilde{\nabla}^\beta$. From Theorem 2, and equation (22) with $\|\operatorname{grad} f\| = 1$, $\operatorname{Hess}_f(\operatorname{grad} f, X) = 0$, for all $X \in \Gamma(TM)$, and $\nabla_{\operatorname{grad} f}\operatorname{grad} f = 0$, we obtain the following

$$\begin{aligned}&\widetilde{R}_\beta(\tau(\widetilde{I}_{\alpha,\beta}), \mathrm{d}\widetilde{I}_{\alpha,\beta}(e_i))\mathrm{d}\widetilde{I}_{\alpha,\beta}(e_i) \\ &\quad = \frac{\alpha-\beta}{\alpha}\Delta f\Big\{R(\operatorname{grad} f, e_i)e_i + (1-\beta)g(R(\operatorname{grad} f, e_i)\operatorname{grad} f, e_i)\operatorname{grad} f\Big\},\end{aligned} \tag{23}$$

from (23) and the definition of Ricci curvature, we get

$$\begin{aligned}\widetilde{R}(\tau(\widetilde{I}_{\alpha,\beta}), \mathrm{d}\widetilde{I}_{\alpha,\beta}(e_1))\mathrm{d}\widetilde{I}_{\alpha,\beta}(e_1) &+ \frac{1}{\alpha}\sum_{i=2}^m \widetilde{R}(\tau(\widetilde{I}_{\alpha,\beta}), \mathrm{d}\widetilde{I}_{\alpha,\beta}(e_i))\mathrm{d}\widetilde{I}_{\alpha,\beta}(e_i) \\ &= \frac{\alpha-\beta}{\alpha^2}\Delta f\Big\{\operatorname{Ricci}(\operatorname{grad} f) \\ &\qquad - (1-\beta)\operatorname{Ric}(\operatorname{grad} f, \operatorname{grad} f)\operatorname{grad} f\Big\}.\end{aligned} \tag{24}$$

Let $i = 1, \ldots, m$, we compute

$$\begin{aligned}
\nabla^{\widetilde{I}_{\alpha,\beta}}_{e_i}\nabla^{\widetilde{I}_{\alpha,\beta}}_{e_i}\tau(\widetilde{I}_{\alpha,\beta})-\nabla^{\widetilde{I}_{\alpha,\beta}}_{\widetilde{\nabla}^{\alpha}_{e_i}e_i}\tau(\widetilde{I}_{\alpha,\beta})&\\
&= \frac{\alpha-\beta}{\alpha}\Big\{\widetilde{\nabla}^{\beta}_{e_i}\widetilde{\nabla}^{\beta}_{e_i}\Delta f\,\mathrm{grad}\,f - \widetilde{\nabla}^{\beta}_{\widetilde{\nabla}^{\alpha}_{e_i}e_i}\Delta f\,\mathrm{grad}\,f\Big\}\\
&= \frac{\alpha-\beta}{\alpha}\Big\{\widetilde{\nabla}^{\beta}_{e_i}\nabla_{e_i}\Delta f\,\mathrm{grad}\,f - \nabla_{\widetilde{\nabla}^{\alpha}_{e_i}e_i}\Delta f\,\mathrm{grad}\,f\Big\}\\
&= \frac{\alpha-\beta}{\alpha}\Big\{\nabla_{e_i}\nabla_{e_i}\Delta f\,\mathrm{grad}\,f - \nabla_{\nabla_{e_i}e_i}\Delta f\,\mathrm{grad}\,f\\
&\quad + (1-\beta)\,\mathrm{Hess}_f(e_i,\nabla_{e_i}\Delta f\,\mathrm{grad}\,f)\,\mathrm{grad}\,f\\
&\quad - (1-\alpha)\,\mathrm{Hess}_f(e_i,e_i)\nabla_{\mathrm{grad}\,f}\Delta f\,\mathrm{grad}\,f\Big\}\,,
\end{aligned} \tag{25}$$

a straightforward calculation shows that

$$\begin{aligned}
\nabla_{e_i}\nabla_{e_i}\Delta f\,\mathrm{grad}\,f - \nabla_{\nabla_{e_i}e_i}\Delta f\,\mathrm{grad}\,f&\\
&= e_i(e_i(\Delta f))\,\mathrm{grad}\,f + 2e_i(\Delta f)\nabla_{e_i}\,\mathrm{grad}\,f\\
&\quad + \Delta f\nabla_{e_i}\nabla_{e_i}\,\mathrm{grad}\,f - (\nabla_{e_i}e_i)(\Delta f)\,\mathrm{grad}\,f\\
&\quad - \Delta f\nabla_{\nabla_{e_i}e_i}\,\mathrm{grad}\,f\,,
\end{aligned} \tag{26}$$

$$\begin{aligned}
(1-\beta)\,\mathrm{Hess}_f(e_i,\nabla_{e_i}\Delta f\,\mathrm{grad}\,f)\,\mathrm{grad}\,f&\\
&= -(1-\beta)\Delta f g(\mathrm{grad}\,f,\nabla_{e_i}\nabla_{e_i}\,\mathrm{grad}\,f)\,\mathrm{grad}\,f\,,
\end{aligned} \tag{27}$$

and

$$\begin{aligned}
-(1-\alpha)\,\mathrm{Hess}_f(e_i,e_i)\nabla_{\mathrm{grad}\,f}\Delta f\,\mathrm{grad}\,f&\\
&= -(1-\alpha)\,\mathrm{Hess}_f(e_i,e_i)(\mathrm{grad}\,f)(\Delta f)\,\mathrm{grad}\,f\,,
\end{aligned} \tag{28}$$

by equations (25)–(28), with $\|\mathrm{grad}\,f\| = 1$, we find that

$$\begin{aligned}
&\nabla^{\widetilde{I}_{\alpha,\beta}}_{e_1}\nabla^{\widetilde{I}_{\alpha,\beta}}_{e_1}\tau(\widetilde{I}_{\alpha,\beta}) - \nabla^{\widetilde{I}_{\alpha,\beta}}_{\widetilde{\nabla}^{\alpha}_{e_1}e_1}\tau(\widetilde{I}_{\alpha,\beta}) + \frac{1}{\alpha}\sum_{i=2}^{m}\Big\{\nabla^{\widetilde{I}_{\alpha,\beta}}_{e_i}\nabla^{\widetilde{I}_{\alpha,\beta}}_{e_i}\tau(\widetilde{I}_{\alpha,\beta}) - \nabla^{\widetilde{I}_{\alpha,\beta}}_{\widetilde{\nabla}^{\alpha}_{e_i}e_i}\tau(\widetilde{I}_{\alpha,\beta})\Big\}\\
&\quad= \frac{\alpha-\beta}{\alpha^2}\Big\{(\alpha-1)\,\mathrm{Hess}_{\Delta f}(\mathrm{grad}\,f,\mathrm{grad}\,f)\,\mathrm{grad}\,f + \Delta^2 f\,\mathrm{grad}\,f\\
&\qquad + 2\nabla_{\mathrm{grad}\,\Delta f}\,\mathrm{grad}\,f + \Delta f\,\mathrm{trace}\,\nabla^2\,\mathrm{grad}\,f\\
&\qquad - (1-\beta)\Delta f g(\mathrm{grad}\,f,\mathrm{trace}\,\nabla^2\,\mathrm{grad}\,f)\,\mathrm{grad}\,f\\
&\qquad - (1-\alpha)\Delta f g(\mathrm{grad}\,f,\mathrm{grad}\,\Delta f)\,\mathrm{grad}\,f\Big\}\,,
\end{aligned} \tag{29}$$

from equations (24), (29), and the following (see [1])

$$\mathrm{trace}\,\nabla^2\,\mathrm{grad}\,f = \mathrm{Ricci}(\mathrm{grad}\,f) + \mathrm{grad}(\Delta f),$$

the identity map $\widetilde{I}_{\alpha,\beta}$ is a proper biharmonic map if and only if

$$\begin{aligned}2\Delta f\,\mathrm{Ricci}(\operatorname{grad} f) &- 2(1-\beta)\Delta f\,\mathrm{Ric}(\operatorname{grad} f, \operatorname{grad} f)\operatorname{grad} f\\ &+ \Delta^2 f \operatorname{grad} f + 2\nabla_{\operatorname{grad}\Delta f}\operatorname{grad} f + \Delta f \operatorname{grad}\Delta f\\ &- (2-\alpha-\beta)\Delta f g(\operatorname{grad} f, \operatorname{grad}\Delta f)\operatorname{grad} f\\ &+ (\alpha-1)\,\mathrm{Hess}_{\Delta f}(\operatorname{grad} f, \operatorname{grad} f)\operatorname{grad} f = 0\,, \end{aligned} \tag{30}$$

with $\alpha \neq \beta$ and $\Delta f \neq 0$, taking its inner product with $\operatorname{grad} f$, we have

$$\begin{aligned}-&\,2(1-\beta)\Delta f\,\mathrm{Ric}(\operatorname{grad} f, \operatorname{grad} f)\\ &= \frac{1-\beta}{\beta}\Delta^2 f - \frac{(1-\beta)(1-\alpha)}{\beta}\,\mathrm{Hess}_{\Delta f}(\operatorname{grad} f, \operatorname{grad} f)\\ &\quad - \frac{(1-\beta)(1-\alpha-\beta)}{\beta}\Delta f g(\operatorname{grad} f, \operatorname{grad}\Delta f)\,. \end{aligned} \tag{31}$$

Theorem 4 follows from (30) and (31). □

Corollary 1. *If $\alpha \neq \beta$, $\|\operatorname{grad} f\| = 1$, $\Delta f = F(f)$, where F is a non-null function on $I \subset \mathbb{R}$, and $\mathrm{Ricci}(\operatorname{grad} f) = \lambda \operatorname{grad} f$, for some smooth function λ on M. Then the identity map $\widetilde{I}_{\alpha,\beta}$ is a proper biharmonic if and only if the function f satisfying the following*

$$2\beta\lambda F(f) + (\alpha+\beta)F(f)F'(f) + \alpha F''(f) = 0\,.$$

According to Corollary 1, we have the following example.

Example 2. Let $M = (0,\infty) \times \mathbb{R}^n$ equipped with the Riemannian metric

$$g = \mathrm{d}t^2 + \frac{\mathrm{d}x_1^2 + \cdots + \mathrm{d}x_n^2}{t}\,,$$

we set $f(t,x) = t$, for all $(t,x) \in M$. We have $\operatorname{grad} f = \partial_t$, $\|\operatorname{grad} f\| = 1$, $\Delta f = -\frac{n}{2t}$ and $\mathrm{Ricci}(\operatorname{grad} f) = -\frac{3n}{4t^2}\partial_t$, so that $F(s) = -\frac{n}{2s}$, for all $s \in I = (0,\infty)$ and $\lambda(t,x) = -\frac{3n}{4t^2}$ for all $(t,x) \in M$. Using the Corollary 1, Then the identity map $\widetilde{I}_{\alpha,\beta}$ is proper biharmonic if and only if $n \neq 4$ and $\alpha = \frac{2n\beta}{n+4}$.

5 Biharmonic curve in $(M, \tilde{g}_\alpha)$

Let $\gamma\colon I \subset \mathbb{R} \to (M,g)$, $t \mapsto \gamma(t)$ be a harmonic curve in a Riemannian manifold (M,g), such that $g(\dot\gamma,\dot\gamma) = 1$, and let f be a smooth function on M. In this section we suppose that the gradient vector of f at $\gamma(t)$ is parallel to the tangent vector $\dot\gamma(t)$. Thus, $(\operatorname{grad} f)_{\gamma(t)} = \rho(t)\dot\gamma(t)$, with $\rho(t) = (f\circ\gamma)'(t)$, for all $t \in I$. Since γ is harmonic we get the following formula

$$(\nabla_{\dot\gamma}\operatorname{grad} f)_t = \rho'(t)\dot\gamma(t)\,, \quad \forall t \in I. \tag{32}$$

We set $\tilde{g}_\alpha = \alpha g + (1-\alpha)\mathrm{d}f \otimes \mathrm{d}f$, where $\alpha \in (0,1)$. We have the following result:

Theorem 5. *The curve $\gamma\colon I \to (M, \tilde{g}_\alpha)$ is biharmonic if and only if the function f satisfying the following*

$$f(\gamma(t)) = \pm \int \sqrt{(at^2+bt+c)^2 - \frac{\alpha}{1-\alpha}}\,\mathrm{d}t\,,$$

where $a, b, c \in \mathbb{R}$, such that $(at^2+bt+c)^2 > \frac{\alpha}{1-\alpha}$, for all $t \in I$.

Proof. By Theorem 1, we have

$$\widetilde{\tau}(\gamma) = \tau(\gamma) + \frac{(1-\alpha)\operatorname{Hess}_f(\dot\gamma,\dot\gamma)}{\alpha + (1-\alpha)\|\operatorname{grad} f\|^2 \circ \gamma}(\operatorname{grad} f)\circ\gamma\,, \tag{33}$$

from the harmonicity condition of γ, and equations (32), (33), we obtain $\widetilde{\tau}(\gamma) = \lambda\dot\gamma$, where

$$\lambda = \frac{(1-\alpha)\rho\rho'}{\alpha + (1-\alpha)\rho^2}\,. \tag{34}$$

Now, the curve $\gamma\colon I \to (M, \tilde{g}_\alpha)$ is biharmonic if and only if

$$\widetilde{R}\Big(\widetilde{\tau}(\gamma), \mathrm{d}\gamma\Big(\frac{\mathrm{d}}{\mathrm{d}t}\Big)\Big)\mathrm{d}\gamma\Big(\frac{\mathrm{d}}{\mathrm{d}t}\Big) + \widetilde{\nabla}^\gamma_{\frac{\mathrm{d}}{\mathrm{d}t}}\widetilde{\nabla}^\gamma_{\frac{\mathrm{d}}{\mathrm{d}t}}\widetilde{\tau}(\gamma) = 0\,, \tag{35}$$

by the property of the curvature tensor, the first term on the left-hand side of (35) is

$$\widetilde{R}\Big(\widetilde{\tau}(\gamma), \mathrm{d}\gamma\Big(\frac{\mathrm{d}}{\mathrm{d}t}\Big)\Big)\mathrm{d}\gamma\Big(\frac{\mathrm{d}}{\mathrm{d}t}\Big) = \lambda\widetilde{R}(\dot\gamma,\dot\gamma)\dot\gamma = 0\,.$$

For the second term on the left-hand side of (35), we compute

$$\begin{aligned}\widetilde{\nabla}^\gamma_{\frac{\mathrm{d}}{\mathrm{d}t}}\widetilde{\tau}(\gamma) &= \widetilde{\nabla}^\gamma_{\frac{\mathrm{d}}{\mathrm{d}t}}\lambda\dot\gamma\\ &= \lambda'\dot\gamma + \lambda\widetilde{\nabla}_{\dot\gamma}\dot\gamma\\ &= (\lambda' + \lambda^2)\dot\gamma\,,\end{aligned} \tag{36}$$

with the same method of (36), we find that

$$\begin{aligned}\widetilde{\nabla}^\gamma_{\frac{\mathrm{d}}{\mathrm{d}t}}\widetilde{\nabla}^\gamma_{\frac{\mathrm{d}}{\mathrm{d}t}}\widetilde{\tau}(\gamma) &= \widetilde{\nabla}^\gamma_{\frac{\mathrm{d}}{\mathrm{d}t}}(\lambda' + \lambda^2)\dot\gamma\\ &= (\lambda'' + 2\lambda\lambda')\dot\gamma + (\lambda' + \lambda^2)\widetilde{\nabla}_{\dot\gamma}\dot\gamma\\ &= (\lambda'' + 3\lambda\lambda' + \lambda^3)\dot\gamma\,.\end{aligned}$$

So, the curve $\gamma\colon I \to (M, \tilde{g}_\alpha)$ is biharmonic if and only if $\lambda'' + 3\lambda\lambda' + \lambda^3 = 0$, that is the function λ is the form $(2at+b)/(at^2+bt+c)$, where $a, b, c \in \mathbb{R}$, such that $at^2+bt+c \neq 0$, for all $t \in I$. Thus, from (34) with $(at^2+bt+c)^2 > \frac{\alpha}{1-\alpha}$, for all $t \in I$, we obtain

$$\rho(t) = \pm\sqrt{(at^2+bt+c)^2 - \frac{\alpha}{1-\alpha}}\,, \quad \forall t \in I. \tag{37}$$

Theorem 5 follows from equation (37), with $\rho = (f\circ\gamma)'$. $\square$

Remark 2. The curve $\gamma\colon I \to (M, \tilde{g}_\alpha)$ is proper biharmonic if and only if there exists $a, b, c \in \mathbb{R}$ such that $a^2 + b^2 > 0$, and for all $i = 1, \dots, m$ ($m = \dim M$), and in any local coordinates (x_i) on M, such that

$$\sum_{j=1}^{m} g^{ij}(\gamma(t)) \frac{\partial f}{\partial x_j}\Big|_{\gamma(t)} = \pm\sqrt{(at^2 + bt + c)^2 - \frac{\alpha}{1-\alpha}\,\frac{\mathrm{d}\gamma^i}{\mathrm{d}t}\Big|_t}, \quad \forall t \in I.$$

Using Theorem 5 and the previous Remark, we can construct many examples for proper biharmonic curves.

Example 3. Let $M = \mathbb{R}^n$ equipped with the Riemannian metric $g = \mathrm{d}x_1^2 + \cdots + \mathrm{d}x_n^2$,

$$f(x) = \frac{2}{3}\sum_{i=1}^{n}(1 + x_i^2)^{\frac{3}{2}}, \quad \forall x = (x_1, \dots, x_n) \in M.$$

For $\alpha = \frac{n}{n+1}$, the curve

$$\gamma\colon I \to (M, \tilde{g}_\alpha)\,, \quad t \mapsto (\frac{t}{\sqrt{n}}, \dots, \frac{t}{\sqrt{n}}),$$

is proper biharmonic.

Acknowledgements

The authors would like to thank the reviewers for their useful remarks and suggestions. The authors are supported by National Agency Scientific Research of Algeria and Laboratory of Geometry, Analysis, Control and Applications, Algeria.

References

[1] P. Baird, A. Fardoun, S. Ouakkas: Conformal and semi-conformal biharmonic maps. *Annals of Global Analysis and Geometry* 34 (4) (2008) 403–414.

[2] P. Baird, D. Kamissoko: On constructing biharmonic maps and metrics. *Annals of Global Analysis and Geometry* 23 (1) (2003) 65–75.

[3] P. Baird, J.C. Wood: *Harmonic morphisms between Riemannian manifolds*. Oxford University Press (2003).

[4] A. Benkartab, A.M. Cherif: New methods of construction for biharmonic maps. *Kyungpook Mathematical Journal* 59 (1) (2019) 135–147.

[5] R. Caddeo, S. Montaldo, C. Oniciuc: Biharmonic submanifolds of $\mathbb{S}^3$. *International Journal of Mathematics* 12 (08) (2001) 867–876.

[6] J. Eells, L. Lemaire: A report on harmonic maps. *Bulletin of the London Mathematical Society* 10 (1) (1978) 1–68.

[7] J. Eells, L. Lemaire: Another report on harmonic maps. *Bulletin of the London Mathematical Society* 20 (5) (1988) 385–524.

[8] J. Eells, J.H. Sampson: Harmonic mappings of Riemannian manifolds. *American Journal of Mathematics* 86 (1) (1964) 109–160.

[9] T. Körpinar, E. Turhan: Tubular surfaces around timelike biharmonic curves in Lorentzian Heisenberg group Heis^3. *Analele Universitatii "Ovidius" Constanta – Seria Matematica* 20 (1) (2012) 431–446.

[10] C. Oniciuc: New examples of biharmonic maps in spheres. *Colloquium Mathematicum* 97 (1) (2003) 131–139.

[11] S. Ouakkas: Biharmonic maps, conformal deformations and the Hopf maps. *Differential Geometry and its Applications* 26 (5) (2008) 495–502.

[12] G.Y. Jiang: 2-harmonic maps and their first and second variational formulas. *Chinese Ann. Math. Ser. A* 7 (4) (1986) 389–402.

[13] B. O'Neill: *Semi-Riemannian geometry with applications to relativity*. Academic Press (1983).

[14] T. Sakai: *Riemannian geometry*. Shokabo, Tokyo (1992). (in Japanese)

Received: 11 January 2019
Accepted for publication: 25 June 2020
Communicated by: Haizhong Li